Progressions for the Common Core State Standards in Mathematics (draft)

26 December 2011

6-7, Ratios and Proportional Relationships

Overview

The study of ratios and proportional relationships extends students' work in measurement and in multiplication and division in the elementary grades. Ratios and proportional relationships are foundational for further study in mathematics and science and useful in everyday life. Students use ratios in geometry and in algebra when they study similar gures and slopes of lines, and later when they study sine, cosine, tangent, and other trigonometric ratios in high school. Students use ratios when they work with situations involving constant rates of change, and later in calculus when they work with average and instantaneous rates of change of functions. An understanding of ratio is essential in the sciences to make sense of quantities that involve derived attributes such as speed, acceleration, density, surface tension, electric or magnetic eld strength, and to understand percentages and ratios used in describing chemical solutions. Ratios and percentages are also useful in many situations in daily life, such as in cooking and in calculating tips, miles per gallon, taxes, and discounts. They also are also involved in a variety of descriptive statistics, including demographic, economic, medical, meteorological, and agricultural statistics (e.g., birth rate, per capita income, body mass index, rain fall, and crop yield) and underlie a variety of measures, for example, in nance (exchange rate), medicine (dose for a given body weight), and technolog51, medicine
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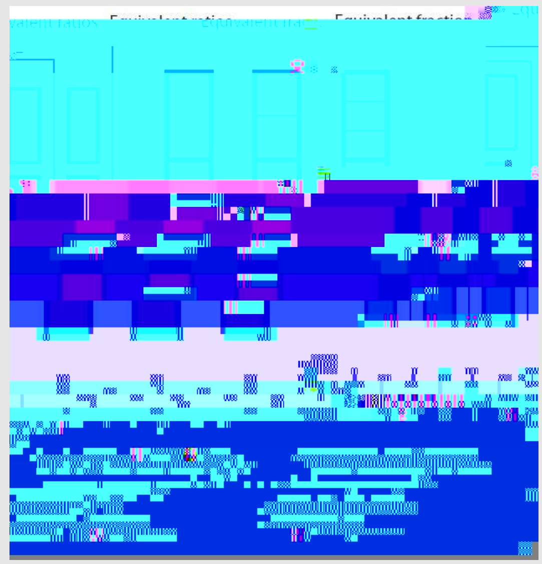
tionship from other types of situations. For example, without further information 2 pounds for a dollar is ambiguous. It may be that pounds and dollars are proportionally related and every two pounds costs a dollar. Or it may be that there is a discount on bulk, so weight and cost do not have a proportional relationship. Thus, recognizing ratios, rates, and proportional relationships involves looking for structure (MP7). Describing and interpreting descriptions of ratios, rates, and proportional relationships involves precise use of language (MP6).

Equivalent ratios versus equivalent fractions



Representing ratios, collections of equivalent ratios, rates, and proportional relationships Because ratios and rates are di erent and rates will often be written using fraction notation in high school, ratio notation should be distinct from fraction notation.

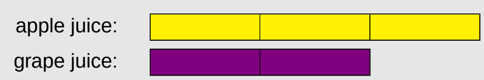
Together with tables, students can also use tape diagrams and double number line diagrams to represent collections of equivalent ratios. Both types of diagrams visually depict the relative sizes of the quantities.

Tape diagrams are best used when the two quantities have the same units. They can be used to solve problems and also to highlight the multiplicative relationship between the quantities.

Double number line diagrams are best used when the quantities have di erent units (otherwise the two diagrams will use di erent length units to represent the same amount). Double number line diagrams can help make visible that there are many, even in nitely many, pairs in the same ratio, including those with rational number entries. As in tables, unit rates appear paired with 1.

A collection of equivalent ratios can be graphed in the coordinate plane. The graph represents a proportional relationship. The unit rate appears in the equation and graph as the slope of the line, and in the coordinate pair with rst coordinate 1.

Representing ratios with tape diagrams



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Grade 6

Representing and reasoning about ratios and collections of equivalent ratios Because the multiplication table is familiar to sixth graders, situations that give rise to columns or rows of a multiplication table can provide good initial contexts when ratios and proportional relationships are introduced. Pairs of quantities in equivalent ratios arising from whole number measurements such as 3 lemons for every $1 or for every 5 cups grape juice, mix in 2 cups peach juice lend themselves to being recorded in a table.[6.RP.3a] Initially, when students make tables of quantities in equivalent ratios, they may focus only on iterating the related quantities by repeated addition to generate equivalent ratios.

As students work with tables of quantities in equivalent ratios (also called ratio tables), they should practice using and understanding ratio and rate language.[6.RP.1,6.RP.2] It is important for students to focus on the meaning of the terms for every, for each, for each 1, and per because these equivalent ways of stating ratios and rates are at the heart of understanding the structure in these tables, providing a foundation for learning about proportional relationships in Grade 7.

Students graph the pairs of values displayed in ratio tables on coordinate axes. The graph of such a collection of equivalent ratios lies on a line through the origin, and the pattern of increases in the table can be seen in the graph as coordinated horizontal and vertical increases.[6.EE.9]

6.RP.3a Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

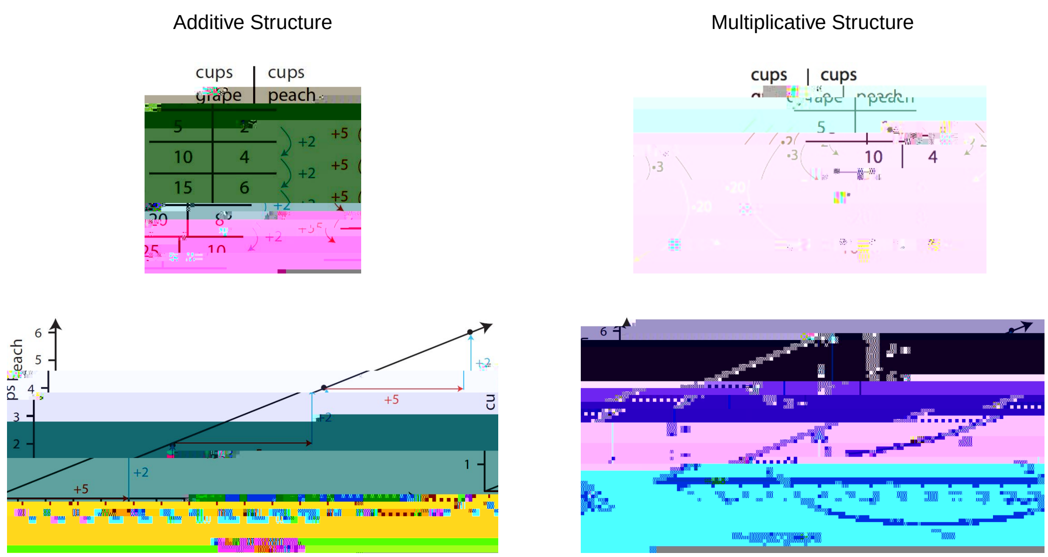
a Make tables of equivalent ratios relating quantities with whole-number measurements, nd missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

6.RP.1 Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities.

6.RP.2 Understand the concept of a unit rate a/b associated with a ratio $a : b$ with $b \neq 0$, and use rate language in the context of a ratio relationship.

6.EE.9 Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.

Showing structure in tables and graphs



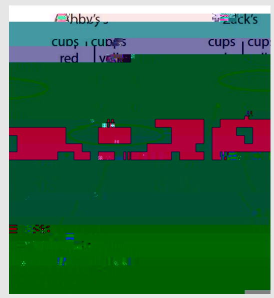
In the tables, equivalent ratios are generated by repeated addition (left) and by scalar multiplication (right). Students might be asked to identify and explain correspondences between each table and the graph beneath it (MP1).

By reasoning about ratio tables to compare ratios,[6.RP.3a] students can deepen their understanding of what a ratio describes in a context and what quantities in equivalent ratios have in common. For example, suppose Abby's orange paint is made by mixing 1 cup red paint for every 3 cups yellow paint and Zack's orange paint is made by mixing 3 cups red for every 5 cups yellow. Students could discuss that all the mixtures within a single ratio table for one of the paint mixtures are the same shade of orange because they are multiple batches of the same mixture. For example, 2 cups red and 6 cups yellow is two batches of 1 cup red and 3 cups yellow; each batch is the same color, so when the two batches are combined, the shade of orange doesn't change. Therefore, to compare the colors of the two paint mixtures, any entry within a ratio table for one mixture can be compared with any entry from the ratio table for the other mixture.

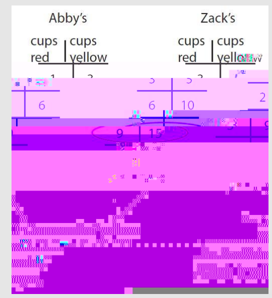
It is important for students to focus on the rows (or columns) of a ratio table as multiples of each other. If this is not done, a common error when working with ratios is to make additive comparisons. For example, students may think incorrectly that the ratios $1:3$ and $3:5$ of red to yellow in Abby's and Zack's paints are equivalent because the difference between the number of cups of red and yellow in both paints is the same, or because Zack's paint could be made from Abby's by adding 2 cups red and 2 cups yellow. The margin shows several ways students could reason correctly to compare the paint mixtures.

Three ways to compare paint mixtures

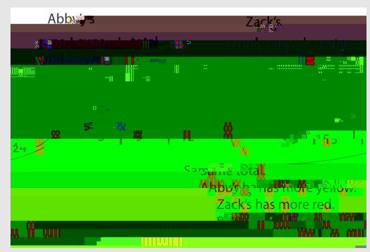
Same amount of red



Same amount of yellow



Same total

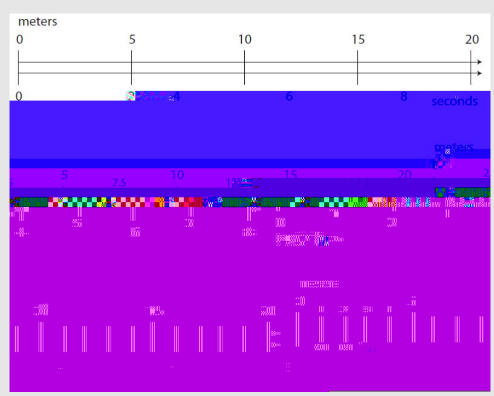


Strategies for solving problems Although it is traditional to move students quickly to solving proportions by setting up an equation, the Standards do not require this method in Grade 6. There are a number of strategies for solving problems that involve ratios. As students become familiar with relationships among equivalent ratios, their strategies become increasingly abbreviated and efficient.

For example, suppose grape juice and peach juice are mixed in a ratio of 5 to 2 and we want to know how many cups of grape juice to mix with 12 cups of peach juice so that the mixture will still be in the same ratio. Students could make a ratio table as shown in the margin, and they could use the table to find the grape juice entry that pairs with 12 cups of peach juice in the table. This perspective allows students to begin to reason about proportions by starting with their knowledge about multiplication tables and by building on this knowledge.

As students generate equivalent ratios and record them in tables, their attention should be drawn to the important role of multiplication and division in how entries are related to each other. Initially, students may fill ratio tables with columns or rows of the multiplication table by skip counting, using only whole number entries, and placing these entries in numerical order. Gradually, students should consider entries in ratio tables beyond those they find by skip counting, including larger entries and fraction or decimal entries. Finding

Double number line diagrams used for situations with different units



Double number lines indicate coordinated multiplying and dividing of quantities. This can also be indicated in tables.

these other entries will require the explicit use of multiplication and division, not just repeated addition or skip counting. For example, if Seth runs 5 meters every 2 seconds, then Seth will run 2.5 meters in 1 second because in half the time he will go half as far. In other words, when the elapsed time is divided by 2, the distance traveled should also be divided by 2. More generally, if the elapsed time is multiplied (or divided) by N, the distance traveled should also be multiplied (or divided) by N. Double number lines can be useful in

Grade 7

In Grade 7, students extend their reasoning about ratios and proportional relationships in several ways. Students use ratios in cases that involve pairs of rational number entries, and they compute associated unit rates. They identify these unit rates in representations of proportional relationships. They work with equations in two variables to represent and analyze proportional relationships. They also solve multi-step ratio and percent problems, such as problems involving percent increase and decrease.

At this grade, students will also work with ratios speci ed by rational numbers, such as $\frac{3}{4}$ cups our for every $\frac{1}{2}$ stick butter.[7.RP.1] Students continue to use ratio tables, extending this use to nding unit rates.

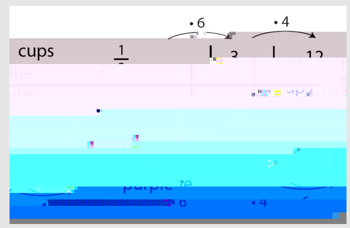
Recognizing proportional relationships Students examine situa-

7.RP.1 Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units.

Ratio problem speci ed by rational numbers: Three approaches

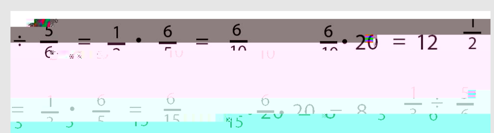
To make Perfect Purple paint mix $\frac{1}{2}$ cup blue paint with $\frac{1}{3}$ cup red paint. If you want to mix blue and red paint in the same ratio to make 20 cups of Perfect Purple paint, how many cups of blue paint and how many cups of red paint will you need?

Method 1



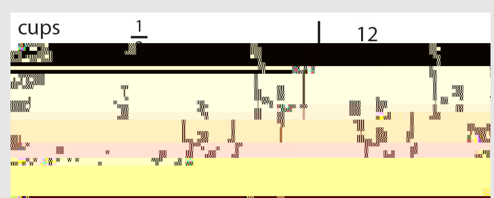
"I thought about making 6 batches of purple because that is a whole number of cups of purple. To make 6 batches, I need 6 times as much blue and 6 times as much red too. That was 3 cups blue and 2 cups red and that made 5 cups purple. Then 4 times as much of each makes 20 cups purple."

Method 2



"I found out what fraction of the paint is blue and what fraction is red. Then I found those fractions of 20 to nd the number of cups of blue and red in 20 cups."

Method 3



Like Method 2, but in tabular form, and viewed as multiplicative comparisons.

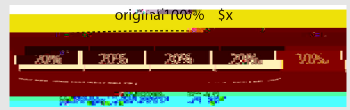
problem statements is more di cult than the rst because of the reversal.

> If a factory produces 5 cans of dog food for every 3 cans of cat food, then when the company produces 600 cans of dog food, how many cans of cat food will it produce?
>
> If a factory produces 5 cans of dog food for every 3 cans of cat food, then how many cans of cat food will the company produce when it produces 600 cans of dog food?

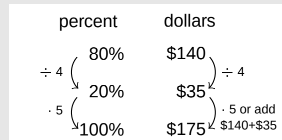
Such problems can be framed in terms of proportional relationships and the constant of proportionality or unit rate, which is obscured by the traditional method of setting up proportions. For example, if Seth runs 5 meters every 2 seconds, he runs at a rate of 2.5 meters per second, so distance d (in meters) and time t (in seconds) are related by $d = 2.5t$. If $d = 100$ then $t = \frac{100}{2.5} = 40$, so he takes 40 seconds to run 100 meters.

7.RP.3 Use proportional relationships to solve multistep ratio and percent problems.

Skateboard problem 1



After a 20% discount, the price is 80% of the original price. So 80% of the original is $140.



“To nd 20% I divided by 4. Then 80% plus 20% is 100%”

Connection to Geometry One new context for proportions at Grade 7 is scale drawings.[7.G.1] To compute unknown lengths from known lengths, students can set up proportions in tables or equations, or they can reason about how lengths compare multiplicatively. Students can use two kinds of multiplicative comparisons. They can apply a scale factor that relates lengths in two di erent gures, or they can consider the ratio of two lengths within one gure, nd a multiplicative relationship between those lengths, and apply that relationship to the ratio of the corresponding lengths in the other gure. When working with areas, students should be aware that areas do not scale by the same factor that relates lengths. (Areas scale by the square of the scale factor that relates lengths, if area is measured in the unit of measurement derived from that used for length.)

Connection to Statistics and Probability Another new context for proportions at Grade 7 is to drawing inferences about a population from a random sample.[7.SP.1] Because random samples can be expected to be approximately representative of the full population, one can imagine selecting many samples of that same size until the full population is exhausted, each with approximately the same chales inferenc-32f(322(same)-3216(selec447(sampla0d [(popefc447(

Using percentages in comparisons

There are 25% more seventh graders than sixth graders in the after-school club. If there are 135 sixth and seventh graders altogether in the after-school club, how many are sixth graders and how many are seventh graders?

"25% more seventh graders than sixth graders means that the number of extra seventh graders is the same as 25% of the sixth graders."

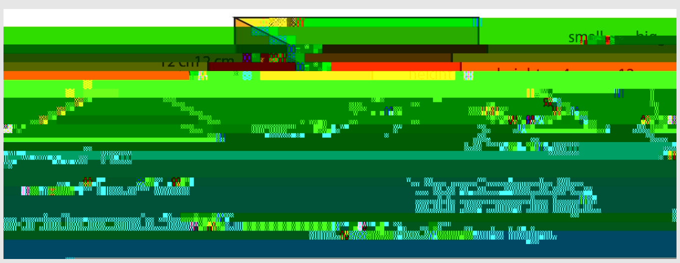
7.G.1 Solve problems involving scale drawings of geometric gures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

Connection to geometry

If the two rectangles are similar, then how wide is the larger rectangle?

Use a scale factor: Find the scale factor from the small rectangle to the larger one:

Use an internal comparison: Compare the width to the height in the small rectangle. The ratio of the width to height is the same in the large rectangle.



Connection to statistics and probability

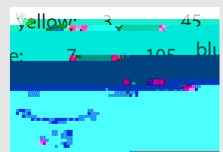
There are 150 tiles in a bin. Some of the tiles are blue and the rest are yellow. A random sample of 10 tiles was selected. Of the 10 tiles, 3 were yellow and 7 were blue. What are the best estimates for how many blue tiles are in the bin and how many yellow tiles are in the bin?

Student 1

yellow:	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
blue:	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
total:	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150

"I gured if you keep picking out samples of 10 they should all be about the same, so I got this ratio table. Out of 150 tiles, about 45 should be yellow and about 105 should be blue."

Student 2



"I also made a ratio table. I said that if there are 15 times as many tiles in the bin as in the sample, then there should be about 15 times as many yellow tiles and 15 times as many blue tiles. $15 \cdot 3 = 45$, so 45 yellow tiles. $15 \cdot 7 = 105$, so 105 blue tiles."

Student 3

30% yellow tiles $\quad 30\% \cdot 150 = \frac{3 \cdot 10}{10 \cdot 10} \cdot 150 = \frac{3}{10} \cdot 15 \cdot 10 = 45$

70% blue tiles $\quad 70\% \cdot 150 = \frac{7 \cdot 10}{10 \cdot 10} \cdot 150 = \frac{7}{10} \cdot 15 \cdot 10 = 105$

"I used percentages. 3 out of 10 is 30% yellow and 7 out of 10 is 70% blue. The percentages in the whole bin should be about the same as the percentages in the sample."

Appendix: A framework for ratio, rate, and proportional relationships

This section presents definitions of the terms ratio, rate, and proportional relationship that are consistent with the Standards and it briefly summarizes some of the essential characteristics of these concepts. It also provides an organizing framework for these concepts. Because many different authors have used ratio and rate terminology in widely differing ways, there is a need to standardize the terminology for use with the Standards and to have a common framework for curriculum developers, professional development providers, and other education professionals to discuss the concepts. This section does not describe how the concepts should be presented to students in Grades 6 and 7.

Definitions and essential characteristics of ratios, rates, and proportional relationships

A ratio is a pair of non-negative numbers, $A : B$, which are not both 0.

When there are A units of one quantity for every B units of another quantity, a rate associated with the ratio $A : B$ is $\frac{A}{B}$ units of the first quantity per 1 unit of the second quantity. (Note that the two quantities may have different units.) The associated unit rate is $\frac{A}{B}$. The term unit rate is the numerical part of the rate; the "unit" is used to highlight the 1 in "per 1 unit of the second quantity." Unit rates should not be confused with unit fractions (which have a 1 in the numerator).

A rate is expressed in terms of a unit that is derived from the units of the two quantities (such as m/s, which is derived from meters and seconds). In high school and beyond, a rate is usually written as

$$\frac{A}{B}\,\frac{\text{units}}{\texttt{units}}$$

where the two different fonts highlight the possibility that the quantities may have different units. In practice, when working with a ratio $A : B$, the rate $\frac{A}{B}$ units per 1 `unit` and the rate $\frac{B}{A}$ `units` per 1 unit are both useful.

The value of a ratio $A : B$ is the quotient $\frac{A}{B}$ (if B is not 0). Note that the value of a ratio may be expressed as a decimal, percent, fraction, or mixed number. The value of the ratio $A : B$ tells how A and B compare multiplicatively; specifically, it tells how many times as big A is as B. In practice, when working with a ratio $A : B$, the value $\frac{A}{B}$ as well as the value $\frac{B}{A}$, associated with the ratio $B : A$, are both useful. These values of each ratio are viewed as unit rates in some contexts (see Perspective 1 in the next section).

Two ratios $A : B$ and $C : D$ are equivalent if there is a positive number, c, such that $C = cA$ and $D = cB$. To check that two ratios are equivalent one can check that they have the same value (because $\frac{cA}{cB} = \frac{A}{B}$), or one can cross-multiply and check that $A \cdot D = B \cdot C$ (because $A \cdot cB = B \cdot cA$). Equivalent ratios have the same unit rate.

A proportional relationship is a collection of pairs of numbers that are in equivalent ratios. A ratio $A : B$ determines a proportional relationship, namely the collection of pairs $(cA; cB)$, for c positive.

With this perspective, if the relationship of the two quantities is represented by an equation $y = cx$, the constant of proportionality, c, can be viewed as the numerical part of a rate associated with the ratio $A : B$.

Second perspective: Ratio as fixed numbers of parts Two quantities which have the same units, are in a ratio of A to B if there is a part of some size such that there are A parts present of the first quantity and B parts present of the second quantity. In other words, two quantities are in a ratio of A to B if there is a positive number c (which could be a rational number), such that there are $A \cdot c$ units of the first quantity and $B \cdot c$ units of the second quantity.

With this perspective, one thinks of a ratio as two pieces. One piece is constituted of A parts, the other of B parts. To create pairs of measurements in the same ratio, one specifies an amount and fills each part with that amount. For example, in a ratio of 3 parts sand to 2 parts cement, each part could be filled with 5 cubic yards, so that there are 15 cubic yards of sand and 10 cubic yards of cement; or each part could be filled with 10 cubic meters, so that there are 30 cubic meters of sand and 20 cubic meters of cement. When describing a ratio from this perspective, the units need not be explicitly stated, as in mix sand and cement in a ratio of 3 to 2. However, the type of quantity must be understood or stated explicitly, as in by volume or by weight.

With this perspective, a ratio $A : B$ has an associated value, A A